Exact Renormalization Scheme for Quantum Anosov Maps

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Abstract

An exact renormalization scheme is introduced for quantum Anosov maps (QAMs) on a torus for general boundary conditions (BCs), whose number is always finite. Given a QAM $\hat{U}$ with k BCs and Planck's constant $\hbar = 2\pi/p$ (p integer), its nth renormalization iterate $\hat{U}^{(n)} = \mathcal{R}^n(\hat{U})$ is associated with k BCs for all n and with a Planck's constant $\hbar^{(n)} = \hbar/k^n$. It is shown that the quasienergy eigenvalue problem for $\hat{U}^{(n)}$ for *all* k BCs is equivalent to that for $\hat{U}^{(n+1)}$ at some *fixed* BCs, corresponding, for $n > 0$, to either strict *periodicity* for kp even or *antiperiodicity* for kp odd. The quantum cat maps are, in general, fixed points of either $\mathcal{R}$ or $\mathcal{R}^2$. The Hannay-Berry results turn out then to be significant also for general BCs.

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Nonintegrable systems whose dynamics can be reduced to a 2D torus in phase space have attracted much attention in the quantum-chaos literature. When quantizing such a system on the torus, the admissible quantum states must satisfy proper boundary conditions (BCs), i.e., they have to be periodic on the torus up to constant Bloch phase factors specified by a Bloch wave vector $\mathbf{w}$. If the Hamiltonian of the system is periodic in phase space, such as, for example, the kicked Harper model [1–4], its classical dynamics can be reduced to the toral phase space of one unit cell of periodicity, and all Bloch wave vectors $\mathbf{w}$ in some Brillouin zone (BZ) are allowed. When studying the quantum-chaos problem for such a system, it is natural and important to consider the sensitivity of the eigenstates to continuous variation of $\mathbf{w}$ in the BZ [1–4]. This sensitivity is usually strong for eigenstates spread over the chaotic region and weak for eigenstates localized on stability islands.

In general, however, the Hamiltonian of a system whose dynamics can be reduced to a torus is *not* periodic in phase space. Simple and well known examples are the purely chaotic, Anosov "cat maps" [5–13], whose Hamiltonians are quadratic in the phase-space variables [11]. When quantizing these systems, it turns out that only a *finite* set of $\mathbf{w}$'s in the BZ is allowed [10,12,13], see Eq. (2) below, but this set *increases* with increasing chaotic instability. For a large class of cat maps, the value $\mathbf{w} = \mathbf{0}$, corresponding to strictly periodic quantum states on the torus, is allowed. This class of maps was first quantized, for $\mathbf{w} = \mathbf{0}$, in the well known work of Hannay and Berry [6]. As a matter of fact, almost all the investigations of the quantum cat maps have been confined to this class with $\mathbf{w} = \mathbf{0}$. Recently [13], the case of antiperiodic BCs (the quantum state assumes values of opposite signs on opposite sides of the torus) has been studied in some detail. The results of Hannay and Berry [6] and of Keating [11] revealed a very atypical feature of quantum cat maps, i.e., the high degeneracy in their spectra, which increases in the semiclassical limit.

Typical spectral properties, fitting generic eigenvalue statistics, are already found by quantizing torus maps that are very slight perturbations of the cat maps [14–16]. According

to Anosov's theorem [5], these maps have essentially the same classical dynamics, in particular they are purely chaotic, as the unperturbed cat maps, which are structurally stable. This ceases to be the case for larger perturbations that cause bifurcations generating elliptic islands [16]. However, the quantum BCs for a perturbed cat map are the same as those of the unperturbed cat map, independently of the size of the perturbation [17], see Eq. (2) below. Because of this reason and to simplify terms, general perturbed cat maps are referred to as Anosov maps in this paper. The importance of these maps is in that they may be viewed as *generic* torus maps on the basis of a general expression for a smooth torus map derived recently [17], see below. Understanding the properties of quantum Anosov maps (QAMs) for general toral BCs is essentially an open problem, since almost all the investigations have been confined to the strict-periodicity case, $\mathbf{w} = \mathbf{0}$.

In this paper, we introduce an exact renormalization scheme for QAMs for general BCs on the torus. We show that the spectrum and eigenstates of a general QAM for *all* BCs, and therefore all its quantum properties, can be fully reproduced from those of the renormalized QAM at some special, *fixed* BCs. Thus, the general BCs are practically "eliminated" by the renormalization. Specifically, consider a QAM given by the evolution operator $\hat{U}$ quantizing a classical Anosov map. Quantization on a torus requires a Planck's constant $\hbar$ to satisfy $2\pi/\hbar = p$, an integer. The finite number of BCs is denoted by k, which depends only on the classical unperturbed cat map. We define a renormalization transformation $\mathcal{R}$ generating by iteration a sequence of QAMs $\hat{U}^{(n)} = \mathcal{R}^n(\hat{U})$ on the same torus. The number of BCs for $\hat{U}^{(n)}$ is k for all n, and $\hat{U}^{(n)}$ is associated with a renormalized Planck's constant $\hbar^{(n)} = \hbar/k^n$. Thus, $\hat{U}^{(n)}$ has $k^n p$ eigenstates at fixed BCs. The quantum cat maps are fixed points of either $\mathcal{R}$ or $\mathcal{R}^2$, so that general $\hat{U}^{(n)}$ or $\hat{U}^{(2n)}$ represent perturbations of a given quantum cat map in its classical limit $n \to \infty$. We then show that the quasienergy eigenvalue problem for $\hat{U}^{(n)}$ for *all* k BCs is equivalent, by a unitary transformation accompanied by a scaling of variables, to that for $\hat{U}^{(n+1)}$ at some *fixed* BCs. The latter can be of four types for $n = 0$ ($\hat{U}^{(0)} = \hat{U}$), but, for $n > 0$, they can be only of two types, i.e., strict *periodicity* for kp even

and *antiperiodicity* for kp odd. Thus, the total (all BCs) spectrum of $\hat{U}^{(n)}$, $n = 0, ..., \ \overline{n} - 1$ ($\overline{n} > 1$), coincides with a fraction $k^{1+n-\overline{n}}$ of the spectrum of $\hat{U}^{(\overline{n})}$ for one of these two types of BCs, and the corresponding eigenstates are related by the transformation above. In particular, the total spectrum of a quantum cat map for $\hbar = 2\pi/p$ coincides with a fraction k^{1-n} of its fixed-BCs spectrum for $\hbar = 2\pi/(k^n p)$, with arbitrary or even $n > 0$. It is interesting to note that the two types of BCs above are precisely those that have been studied in detail in the literature [6,7,10–16], so that several results, in particular those of Hannay and Berry [6] for the quantum cat maps, turn out now to be significant also for general BCs.

We denote by $(u, \ v)$ the phase-space variables, $[\hat{u}, \ \hat{v}] = i\hbar$, and we assume that the classical dynamics can be reduced to a $2\pi \times 2\pi$ torus T^2, where it is described by an Anosov map M. In general, a smooth torus map M can be expressed uniquely as the composition of two maps, $M = M_A \circ M_1$ [17]. Here M_A is a cat map, $M_A(\mathbf{z}) = A \cdot \mathbf{z} \mod 2\pi$, where $\mathbf{z}$ is the column vector $(u, \ v)^T$ and A is a 2×2 integer matrix with $\det(A) = 1$; by "Anosov" we just mean that $|\mathrm{Tr}(A)| > 2$, a condition generically satisfied by A. The map M_1 is defined by $M_1(\mathbf{z}) = \mathbf{z} + \mathbf{F}(\mathbf{z}) \mod 2\pi$, where $\mathbf{F}(\mathbf{z})$ is a 2π-periodic vector function of $\mathbf{z}$. The QAM corresponding to $M = M_A \circ M_1$ is the unitary operator $\hat{U} = \hat{U}_A \hat{U}_1$ [17], where $\hat{U}_A$ is the "quantum cat map", whose u representation is [6]

$$\langle u_2 | \hat{U}_A | u_1 \rangle_\hbar = \left(\frac{1}{2\pi i \hbar A_{1,2}} \right)^{1/2} \exp \left[\frac{i}{2\hbar A_{1,2}} \left(A_{1,1} u_1^2 - 2 u_1 u_2 + A_{2,2} u_2^2 \right) \right] \tag{1}$$

with $\hbar = 2\pi/p$ (p integer), and $\hat{U}_1$ is the quantization of the map M_1. We shall assume that M_1 is the map for a Hamiltonian which is periodic in phase space with unit cell T^2. As shown in Ref. [17], this is the case if and only if $\int_{T^2} \mathbf{F}(\mathbf{z}) \, d\mathbf{z} = \mathbf{0}$. Then $\hat{U}_1$ is the one-step evolution operator for the Weyl quantization of this Hamiltonian and is a periodic operator function $\hat{U}_1(\hat{\mathbf{z}}; \ \hbar)$, representable by a well defined Fourier expansion. The toral quantum states must be simultaneous eigenstates of the commuting phase-space translations on T^2, $\hat{D}_1 = \exp(ip\hat{u})$ and $\hat{D}_2 = \exp(-ip\hat{v})$; the corresponding eigenvalues are $\exp(ipw_1)$ and $\exp(-ipw_2)$, where $(w_1, \ w_2)^T = \mathbf{w}$ is the Bloch wave vector specifying the toral BCs

[2]. An eigenstate $\Psi_{\mathbf{w}}$ of $\hat{D}_1$ and $\hat{D}_2$ can be an eigenstate of $\hat{U}$ only for those values of $\mathbf{w}$ in the Brillouin zone (BZ: $0 \leq w_1,\ w_2 < 2\pi/p$) satisfying the equation [17]

$$A \cdot \mathbf{w} = \mathbf{w} + \pi \mathbf{y} \bmod 2\pi/p, \tag{2}$$

where $\mathbf{y} \equiv (A_{1,1}A_{1,2},\ A_{2,1}A_{2,2})^T$. We write the general solution of Eq. (2) as follows:

$$\mathbf{w} = (2\pi/p)B \cdot (\mathbf{r} + pE^{-1} \cdot \mathbf{y}/2) \bmod 2\pi/p, \tag{3}$$

where $B = (A - I)^{-1}E$, I is the identity matrix, E is an arbitrary integer matrix with $\det(E) = \pm 1$, and $\mathbf{r}$ is an integer vector labeling the solutions. There are precisely $k = |\det(B^{-1})| = |2 - \mathrm{Tr}(A)|$ distinct vectors (3), as the number of fixed points of M_A [11], forming a *lattice* in the BZ. We denote by $\mathcal{S}$ the space of states $\Psi_{\mathbf{w}}$ for all these k values of $\mathbf{w}$. The subspace $\mathcal{S}_{\mathbf{w}}$ of $\mathcal{S}$ with a fixed value of $\mathbf{w}$ is p-dimensional, i.e., it is spanned by a basis of p independent states [2,4], whose general expression in the u representation is [18]

$$\Psi_{b,\mathbf{w}}(u) = \sum_{m=0}^{p-1} \phi_b(m;\ \mathbf{w}) \sum_{l=-\infty}^{\infty} e^{ilpw_2} \delta(u - w_1 - 2\pi m/p - 2\pi l), \tag{4}$$

where $b = 1, ...,\ p$. Such a basis is formed, naturally, by the p eigenstates of $\hat{U}$ at fixed $\mathbf{w}$.

We now introduce the torus T_B^2, defined by the vectors $\mathbf{R}_j = 2\pi k(B_{1,j},\ B_{2,j})^T$, $j = 1,\ 2$; kB has integer entries and T_B^2 contains precisely k tori T^2. Since $B^{-1}AB = E^{-1}AE$ is an integer matrix, the superlattice with unit cell T_B^2 is invariant under A, so that the map M modulo T_B^2, denoted by $M^{(B)} = M_A^{(B)} \circ M_1^{(B)}$, is well defined. To continue, we shall first work out in detail the case of $\mathrm{Tr}(A) < -2$, choosing $E = I$, so that $[A,\ B] = 0$ and $\det(B) > 0$. We shall then specify the changes to be made in the case of $\mathrm{Tr}(A) > 2$. Let us perform the linear transformation of variables

$$\mathbf{z} = kB \cdot \mathbf{z}' = \sqrt{k}C \cdot \mathbf{z}', \tag{5}$$

where $C = \sqrt{k}B$ and $\mathbf{z}' = (u',\ v')^T$. Eq. (5) is the combination of a linear canonical transformation $[\det(C) = +1]$ with a scaling by a factor $\sqrt{k}$. Using $[A,\ B] = 0$, it is easy

to check that the map $M^{(B)}$ above is transformed by (5) into the map $M' = M'_A \circ M'_1$ on T^2 in the $\mathbf{z}'$ variables, where M'_A is the ordinary cat map, $M'_A(\mathbf{z}') = A \cdot \mathbf{z}' \bmod 2\pi$, and $M'_1(\mathbf{z}') = (kB)^{-1} \cdot M_1^{(B)}(\mathbf{z} = kB \cdot \mathbf{z}')$. The renormalization transformation $\mathcal{R}_c$ in the classical case is then defined by $M'(\mathbf{z}') = \mathcal{R}_c[M(\mathbf{z})]$. Clearly, the cat maps ($M = M_A$) are fixed points of $\mathcal{R}_c$, i.e., $M'(\mathbf{z}' = \mathbf{z}) = M(\mathbf{z})$.

The quantum version of (5) implies that $[\hat{u}',\ \hat{v}'] = i\hbar'$, where $\hbar' = \hbar/k = 2\pi/p'$, $p' \equiv kp$. The quantization $\hat{U}'$ of $M'(\mathbf{z}')$ is simply $\hat{U}$ expressed in terms of $(\hat{\mathbf{z}}',\ \hbar')$ and acting on the space of the simultaneous eigenstates of the phase-space translations on T^2 in the $\mathbf{z}'$ variables, $\hat{D}'_1 = \exp(ip'\hat{u}')$ and $\hat{D}'_2 = \exp(-ip'\hat{v}')$. It is easy to show that

$$\hat{D}'_{j+1} = \hat{D}(\mathbf{R}_j) = (-1)^{pk^2 B_{1,j} B_{2,j}} \hat{D}_1^{kB_{2,j}} \hat{D}_2^{kB_{1,j}} \tag{6}$$

($j = 1,\ 2,\ \hat{D}'_3 \equiv \hat{D}'_1$), where $\hat{D}(\mathbf{R}_j)$ are precisely the Weyl-Heisenberg phase-space translations on T_B^2. By expressing $\hat{U} = \hat{U}_A \hat{U}_1$ in terms of $(\hat{\mathbf{z}}',\ \hbar')$, using also the theory of linear quantum canonical transformations [19], we obtain the expected result $\hat{U}' = \hat{U}'_A \hat{U}'_1$. Here the u' representation of $\hat{U}'_A$ is given by (1) with u and $\hbar$ replaced by u' and $\hbar'$, respectively, and $\hat{U}'_1$ is the operator function $\hat{U}'_1(\hat{\mathbf{z}}';\ \hbar') = \hat{U}_1(\hat{\mathbf{z}} = kB \cdot \hat{\mathbf{z}}';\ \hbar = k\hbar')$ [the function $\hat{U}_1(\hat{\mathbf{z}};\ \hbar)$ was defined above]. The renormalization transformation $\mathcal{R}$ is then defined by $\hat{U}' = \mathcal{R}(\hat{U})$. By iterating $\mathcal{R}$, one obtains a sequence of QAMs $\hat{U}^{(n)} = \mathcal{R}^n(\hat{U})$ on T^2, associated with the Planck's constants $\hbar^{(n)} = 2\pi/p^{(n)}$, $p^{(n)} \equiv k^n p$. The quantum cat maps ($\hat{U} = \hat{U}_A$) are fixed points of $\mathcal{R}$, i.e., $\langle u'_2 = u_2|\hat{U}'|u'_1 = u_1\rangle_{\hbar'=\hbar} = \langle u_2|\hat{U}|u_1\rangle_{\hbar}$. General $\hat{U}^{(n)} = \hat{U}_A^{(n)} \hat{U}_1^{(n)}$ represent perturbations of the quantum cat map $\hat{U}_A$ in its classical limit $\hbar^{(n)} \to 0$ ($n \to \infty$). The perturbation $\hat{U}_1^{(n)}(\hat{\mathbf{z}}^{(n)};\ \hbar^{(n)})$ is periodic in $\hat{\mathbf{z}}^{(n)}$ with a unit cell $(kB)^{-n} \cdot T^2$, which is k^n times smaller than T^2.

The renormalized Bloch wave vector $\mathbf{w}^{(n)}$ assumes again k values given by Eq. (3) with p replaced by $p^{(n)}$. Consider first $n = 1$. The space of states $\Psi'_{\mathbf{w}'}$ for all the k values of $\mathbf{w}'$ will be denoted by $\mathcal{S}'$. We now show that the original space $\mathcal{S}$ coincides with the subspace

$\mathcal{S}'_{\mathbf{w}'_0}$ of $\mathcal{S}'$ associated with a particular allowed value $\mathbf{w}'_0$. Thus, $\mathcal{S}'$ is k times larger than $\mathcal{S}$. To show this, let us apply $\hat{D}'_j$, $j = 1,\ 2$, on a state $\Psi_{\mathbf{w}}$ of $\mathcal{S}$. Using (3), (6), and the fact that $\hat{D}_j\Psi_{\mathbf{w}} = \exp[i(-1)^{j+1}pw_j]\Psi_{\mathbf{w}}$, $j = 1,\ 2$, we obtain

$$\hat{D}'_j\Psi_{\mathbf{w}} = (-1)^{pA_{j,j+1}}\Psi_{\mathbf{w}} \tag{7}$$

$(A_{2,3} \equiv A_{2,1})$. Rel. (7) means that *all* $\Psi_{\mathbf{w}}$ in $\mathcal{S}$ are eigenstates of $\hat{D}'_j$, $j = 1,\ 2$, associated with the *same* renormalized Bloch wave vector $\mathbf{w}'_0$. The latter can assume only four values, depending on the matrix A, see Table 1.

	k even	k odd
p even	$\mathbf{w}'_0 = \mathbf{0}$	$\mathbf{w}'_0 = \mathbf{0}$
p odd	$\mathbf{w}'_0 = \left(\frac{A_{1,2}\pi}{p'},\ \frac{A_{2,1}\pi}{p'}\right)^T \bmod \frac{2\pi}{p'}$	$\mathbf{w}'_0 = \left(\frac{\pi}{p'},\ \frac{\pi}{p'}\right)^T$

Table 1.

It is easy to show that $\mathbf{w}'_0$ is indeed an allowed value of $\mathbf{w}'$ in all four cases. Thus, $\mathcal{S}'_{\mathbf{w}'_0}$ includes $\mathcal{S}$, but since both $\mathcal{S}'_{\mathbf{w}'_0}$ and $\mathcal{S}$ are kp-dimensional, they coincide. This completes the proof. Now, by the definition above of $\hat{U}'$, the restriction of $\hat{U}'$ to $\mathcal{S}'_{\mathbf{w}'_0} = \mathcal{S}$ is just $\hat{U}$. The kp eigenstates of $\hat{U}$ for all k BCs are then precisely the p' eigenstates of $\hat{U}'$ associated with the value of $\mathbf{w}'_0$ in Table 1. When referred to $\hat{U}'$, however, these eigenstates should be expressed in a representation based on the operator $\hat{\mathbf{z}}'$. If the kp eigenstates of $\hat{U}$ are $\Psi_{b,\mathbf{w}}(u)$ in the u representation, see (4), their u' representation will be obtained by applying to $\Psi_{b,\mathbf{w}}(u)$ the unitary transformation corresponding to a linear canonical transformation [19] with matrix C, after scaling u' by a factor $\sqrt{k}$. The eigenstates of $\hat{U}'$ for $\mathbf{w}' = \mathbf{w}'_0$ are thus given by

$$\Psi'_{b',\mathbf{w}'_0}(u') = \left(\frac{p}{4\pi^2 B_{1,2}}\right)^{1/2} \int_{-\infty}^{\infty} du \exp\left[\frac{-ip}{4\pi B_{1,2}}\left(kB_{1,1}u'^2 - 2u'u + B_{2,2}u^2\right)\right]\Psi_{b,\mathbf{w}}(u), \tag{8}$$

where $b' = b'(b,\ \mathbf{w})$ takes precisely all its p' values when b and $\mathbf{w}$ take all their p and k values, respectively; conversely, $\Psi_{b,\mathbf{w}}(u)$ can be fully reproduced from $\Psi'_{b',\mathbf{w}'_0}(u')$ by inverting Rel. (8) and determining $\mathbf{w}$ by applying $\hat{D}_1$ and $\hat{D}_2$ on $\Psi_{b,\mathbf{w}}(u)$. If the quasienergies of $\hat{U}$

are $\omega_b(\mathbf{w})$, those of $\hat{U}'$ for $\mathbf{w}' = \mathbf{w}'_0$ are $\omega'_{b'(b,\mathbf{w})}(\mathbf{w}'_0) = \omega_b(\mathbf{w})$. The latter relation and Rel. (8) show the equivalence between the quasienergy eigenvalue problem for $\hat{U}$ for all k BCs and that for $\hat{U}'$ at the fixed BCs given by $\mathbf{w}' = \mathbf{w}'_0$.

The generalization of these results to $n > 1$ is straightforward. The fixed BCs for $\hat{U}^{(n)}$ are determined from Table 1 with p, p', and $\mathbf{w}'_0$ replaced by $p^{(n-1)}$, $p^{(n)}$, and $\mathbf{w}_0^{(n)}$, respectively. However, since $p^{(n-1)} = k^{n-1}p$ is always even when k is even, $\mathbf{w}_0^{(n)}$ can take now *only two* values, $\mathbf{w}_0^{(n)} = \mathbf{0}$ (for kp even) and $\mathbf{w}_0^{(n)} = (\pi/p^{(n)},\ \pi/p^{(n)})$ (for kp odd), corresponding to strictly periodic and antiperiodic BCs, respectively. The eigenstates of $\hat{U}^{(n)}$ for $\mathbf{w}^{(n)} = \mathbf{w}_0^{(n)}$ are connected with those of $\hat{U}^{(n-1)}$ for all k BCs by a relation analogous to Rel. (8). The quasienergies are related by $\omega_{b^{(n)}}^{(n)}(\mathbf{w}_0^{(n)}) = \omega_{b^{(n-1)}}^{(n-1)}(\mathbf{w}^{(n-1)})$. Thus, the spectrum and eigenstates of $\hat{U}, ...,\ \hat{U}^{(n-1)}$ for all BCs can be fully reproduced from those of $\hat{U}^{(n)}$ for $\mathbf{w}^{(n)} = \mathbf{w}_0^{(n)}$.

The case of $\mathrm{Tr}(A) > 2$ can be treated similarly only if the integer matrix E in $B = (A - I)^{-1}E$ can be chosen so that $[A,\ E] = 0$ and $\det(E) = -1$. Then one has again $[A,\ B] = 0$ and $\det(C) = +1$ in (5), leading to the same results as above. If $A = K^{2l}$, where K is any integer matrix with $\det(K) = -1$ and l is an integer, one can choose $E = K$; see also the example below. In general, an integer matrix E having the properties above does not exist, and we then make the simple choice $E_{1,1} = E_{2,2} = 0$, $E_{1,2} = E_{2,1} = 1$. While $[A,\ E] \neq 0$, this introduces only "minimal" changes which do not affect the main general results. In all the expressions and equations involving the $n = 1$ renormalized quantities, including in Table 1, A is replaced by $A' = E^{-1}AE$. In the next renormalization, the matrix $B' = (A' - I)^{-1}E$ is used instead of B, starting from Eq. (3), and in all the expressions and equations involving the $n = 2$ quantities A is left unchanged, since $A^{(2)} = E^{-1}A'E = A$. In general, A is replaced by A' only for n odd and B' is then used instead of B in the next, $(n + 1)$th renormalization. Thus, for the quantum cat maps, the dependence of $\hat{U}^{(n)}$ on $(\hat{\mathbf{z}}^{(n)},\ \hbar^{(n)})$ is the same for values of n differing by an even integer and is different otherwise

due to the replacement of A by A'. These maps are therefore fixed points of $\mathcal{R}^2$.

As a simple example, we consider the case when the vectors (3) form a square lattice in the BZ, $\mathbf{w} = 2\pi(r_1,\ r_2)^T/(gp)$, $r_1,\ r_2 = 0, ...,\ g-1$, g integer. This will be the case only if $p\mathbf{y}/2$ in (3) is an integer vector and the matrix E can be chosen so that $B = (A-I)^{-1}E = I/g$. This implies that $A = I + gE$. It is easy to show that the latter relation is satisfied if and only if $A = sE^2$ with $\mathrm{Tr}(E) = sg$, where $s = \pm 1$ is the sign of $\mathrm{Tr}(A)$ [or of $-\det(E) = s$]. Using these conditions, one can easily find matrices A for any g. Note that in this case $\det(B) = g^{-2} > 0$, independently of s, so that the quantum cat maps are always fixed points of $\mathcal{R}$. The transformation (5) is simply $\mathbf{z} = g\mathbf{z}'$, and the eigenstates $\Psi'_{b',\mathbf{w}'_0}(u')$ of $\hat{U}'$ can be easily determined, without using (8), by just substituting $u = gu'$ in (4). After rearranging terms, we find that $\Psi'_{b',\mathbf{w}'_0}(u')$ is given by the expression in Eq. (4) with all the quantities replaced by their primed counterparts and $\mathbf{w}' = \mathbf{w}'_0 = \mathbf{0}$ (in fact, since $p\mathbf{y}/2$ is an integer vector, kp must be even, implying that $\mathbf{w}_0^{(n)} = \mathbf{0}$ for all n). For given b and $\mathbf{w}$, an expansion coefficient $\phi_{b'(b,\mathbf{w})}(m';\ \mathbf{w}'_0)$, $m' = 0, ...,\ p'-1$, is nonzero only if there exists an integer pair $(m,\ l)$, $m = 0, ...,\ p-1$, $l = 0, ...,\ g-1$, solving the Diophantine equation $pgl + gm + r_1 = m'$. The solution is then unique and $\phi_{b'}(m';\ \mathbf{w}'_0) = \exp(2\pi i l r_2/g)\phi_b(m;\ \mathbf{w})$, associated with a "sparse" expansion. In particular, for $p = 1$, i.e., $\hbar' = 2\pi/g^2$, we can choose $\phi_b(m;\ \mathbf{w}) = 1$, and the only nonzero coefficients are $\phi_{b'}(m';\ \mathbf{w}'_0) = \exp(2\pi i l r_2/g)$ with $m' = gl + r_1$. This result can be obtained directly by applying the methods in Ref. [10], where $2\pi/\hbar =$ square integer was assumed, to the case of $A = I + gE$.

In conclusion, the results in this paper provide a first understanding of basic spectral properties of QAMs for general BCs on a torus. The QAMs may be viewed as *generic* quantum torus maps, with the only restriction $|\mathrm{Tr}(A)| > 2$ on their associated matrix A. For given $\hbar = 2\pi/p$, this matrix determines *completely* the general BCs, whose number, k, is always *finite*. This is the basis for the renormalization scheme introduced in this paper. Thus, this scheme cannot be applied to the special torus maps with $\mathrm{Tr}(A) = 2$ (in particular,

the periodic maps with $A = I$, such as the kicked Harper map [1–4]), since $k = \infty$ for them. There is a useful freedom in the definition of the renormalization transformation $\mathcal{R}$, due to the arbitrariness in the choice of the integer matrix E in (3), with $\det(E) = -\mathrm{sgn}[\mathrm{Tr}(A)]$. For $\mathrm{Tr}(A) < -2$ and for at least a large class of matrices A with $\mathrm{Tr}(A) > 2$, E can be chosen so that $[A,\ E] = 0$. Then, the iterates $\hat{U}^{(n)}$ of a QAM $\hat{U}$ under $\mathcal{R}$ are perturbations of the *same* quantum cat map $\hat{U}_A$ with a Planck's constant $\hbar^{(n)} = \hbar/k^n$ tending to the classical limit as $n \rightarrow \infty$. If there exists no E satisfying $[A,\ E] = 0$ for $\mathrm{Tr}(A) > 2$, E can be chosen so that $\hat{U}_A$ is a fixed point of $\mathcal{R}^2$. The main general result, however, is the same for all A: The spectrum and eigenstates of $\hat{U}, ...,\ \hat{U}^{(n-1)}$ for *all* BCs can be fully reproduced from those of $\hat{U}^{(n)}$ at some special, *fixed* BCs. The general BCs are then practically "eliminated" by $\mathcal{R}$ up to an arbitrarily high order n. For $n > 1$, the fixed BCs can be only of two types: strict periodicity for kp even and antiperiodicity for kp odd. Thus, several previous results [6,7,10–16] for these special BCs turn out now to be significant also for general BCs. In addition, some results that can be obtained by previous methods at fixed BCs, such as those in Ref. [10], may now be understood better and derived in a simpler way in the general BCs framework, as in the example above. Finally, the renormalization scheme may be used to study the sensitivity of the eigenstates to variations in the BCs and to derive new exact results concerning, for example, the influence of the general BCs on the spectral degeneracies in the quantum cat maps and on the removal of these degeneracies by small perturbations.

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